Two-Stream Radiative Transfer in FATES - Technical Documentation

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1 Introduction and Assumptions

The conservation equations for the two-stream radiation approximation follow those presented in Sellers (1985) and Bonan (1996). At is core, the mathematical model assumes that the scattering media is homogeneous and continuous in both the vertical and horizontal plane and infinite in the horizontal plane. Radiation intensity is assumed to be uniform in the horizontal plane, and varies over the vertical depth of the scattering media.

The radiation scattering model described here makes some key extensions and breakages from these original assumptions. Firstly, while scattering inside an element is assumed to be continuous and homogenous, we will very often place elements in sequence vertically, where the boundary conditions (i.e. the radiative intensity of their upper and lower edges) balance with another element on its border. And secondly, we allow for the breaking from the assumption of a horizontally infinite plane (if desired), which enables the scattering of radiation through media with different properties in parallel with each other. This enables the representation of the existing fates canopy scattering geometries, such as those formed via the Perfect Plasticity Approximation (Purves et al., 2008). Part of the discussion in this document addresses the affects of breaking this assumption.

2 Governing Conservation Equations and Method of Solution

Here $R_{dn(v)}$ and $R_{up(v)}$ are the down-welling and up-welling diffuse radiation [W m^{-2}] respectively, and they vary over the independent coordinate v which is the integrated amount of media encountered going from the top of the scattering element downward (i.e. total vegetation area index [m^2 m^{-2}]). This uses the following constants: R_{b0} [W m^{-2}] is the beam radiation incident at the top of the scattering media, κ_d and κ_b are the optical depth per unit scattering media for diffuse and beam radiation respectively, ω is the scattering coefficient (ie material reflectance) of the scattering media, and β_d and β_b are the back-scatter fractions of diffuse and beam radiation respectively.

A note about symbols and subscripts: Subscript d refers to "diffuse" not "down", subscript b refers to "beam". Up and down conventions use subscripts up and dn.

For the most part, the diffuse scattering parameters depend on the composition of the media, or in other words, the geometry and composition of the vegetation canopy, whereas the beam scattering constants are dependant on the canopy composition and zenith angle as well.

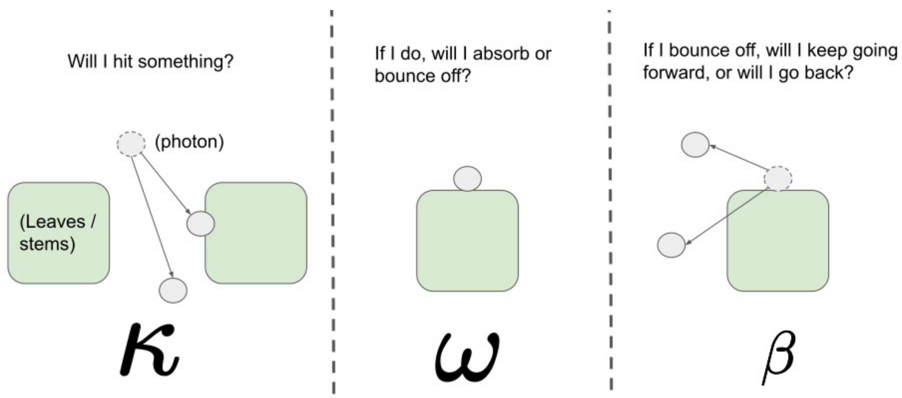

Figure 1: Visualization of the roles the scattering constants play in the two stream system.

The conservation equations are defined as follows:

$$\frac{dR_{dn}}{dv} = \overbrace{-\kappa_d(1-(1-\beta_d)\omega)R_{dn}}^{\text{self sink}} + \overbrace{\kappa_d\beta_d\omega R_{up}}^{\text{up source}} + \overbrace{\kappa_b\omega(1-\beta_b)R_{b0}e^{-\kappa_b v}}^{\text{beam source}}$$

$$-\frac{dR_{up}}{dv} = \overbrace{-\kappa_d(1-(1-\beta_d)\omega)R_{up}}^{\text{self sink}} + \overbrace{\kappa_d\beta_d\omega R_{dn}}^{\text{down source}} + \overbrace{\kappa_b\omega\beta_b R_{b0}e^{-\kappa_b v}}^{\text{beam source}} \tag{1}$$

Each equation has two source terms. One term captures how the opposite stream is converted into the current stream. The other source is where the beam term is scattered into the current stream. The sink term is the rate at which the current stream is either absorbed by the media or re-scattered in the other direction. Note that for the upwelling radiation equation, the sign on the differential is swapped. This is simply because the direction of the upwelling radiation is in the negative v direction.

The generalized analytical solution to Equation 1 is provided in a simplified form in Equation 2. The terms A_{up}, A_{dn}, B_1, B_2 and a represent different complex combinations of the original coefficients κ, β and ω (for readability). The terms λ are solved via a linear system of equations that utilize the boundary conditions at the top of the canopy.

$$\begin{aligned} R_{up(v)} &= A_{up}e^{-\kappa_b v} + B_1 e^{av}\lambda_1 + B_2 e^{-av}\lambda_2 \\ R_{dn(v)} &= A_{dn}e^{-\kappa_b v} + B_2 e^{av}\lambda_1 + B_1 e^{-av}\lambda_2 \end{aligned} \tag{2}$$

There have been several papers published that find solutions to the two-stream. We most closely followed the methods described in Liou (2002). These methods were also adapted by Longo et al. (2019), which are nearyl identical to the mathematical manipulations that describe the formulation of the generalized analytical solution (with the main exception being, that methodology adhered strictly to a system of serial scattering elements, and no parallel elements).

2.1 Method of Solution

The analytical solution shown in Equation 2 was achieved by performing the following manipulations on the conservation equations (Equation 1):

1. re-casting the dependent variables, up and downwelling diffuse radiation, as their difference R_- and their addition R_+
2. calculating the second order differential of these two terms ($\frac{d^2R_-}{dv^2}$ and $\frac{d^2R_+}{dv^2}$), which decouples them from each other
3. deriving a generalized analytical solution for each second order equation with unknown constants
4. reducing the number of unknown constants by combining the analytical solution with the transformed conservation equation
5. translating the analytical solution with reduced constants back to the original up/down coordinate

These steps are carried out in explicit detail in the Appendix, Section 6.1.

3 Solving as a Linear System for Parallel and Serial Elements

As mentioned in Section 1, the two-stream scattering defined above in Equation 2, represent scattering in continuous media with homogeneous scattering coefficients. However, the optical properties and scattering media density of real vegetation canopies vary in vertical and horizontal space.

To enable scattering over media where the scattering coefficients (i.e. different orientation angles, scattering albedos, backscatter and/or unit optical depth) change in space, one method is create a system of discrete scattering elements that are internally homogenous and continuous, yet pass radiation intensity to each other as boundary conditions in a system of equations.

For FATES, we use a Perfect Plasticity Approximation (PPA), which results in the canopy being partitioned in horizontal space by functional types, and vertical space by canopy layer.

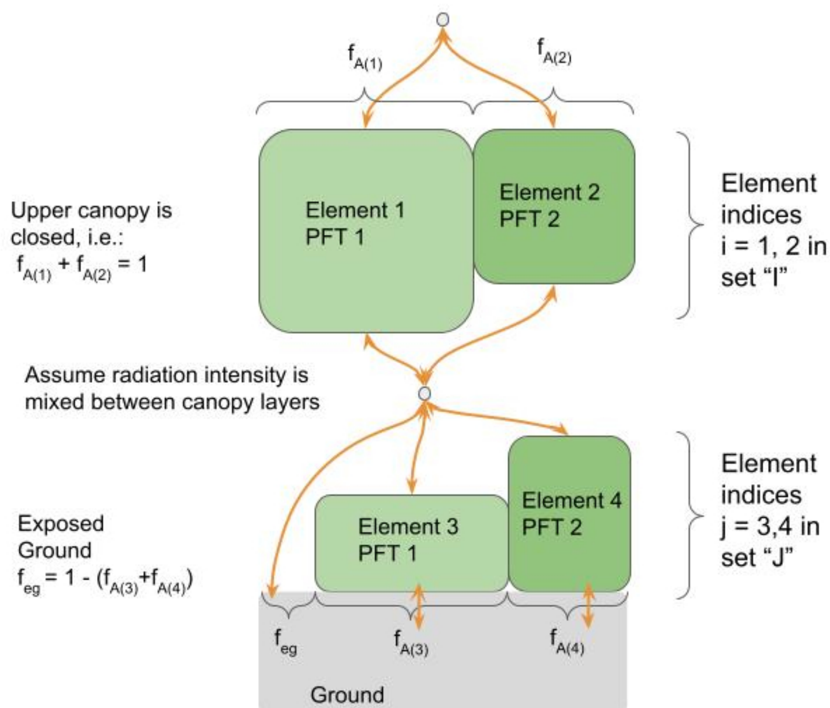

Figure 2: An example diagram of how scattering elements interconnect for two-stream radiation in a FATES assembled vegetation canopy using the Perfect Plasticity Approximation. This example assumes the canopy is compsed of two generic plant functional types (PFT 1 and 2), where both functional types inhabit an upper story (canopy) and lower story. By PPA definition, any layer above the understory is fully closed, and any layer at the ground will have some non-zero amount of exposed soil. The orange lines represent how downwelling and upwelling diffuse radiation flow from the atmosphere to the scattering elements, and then on to scattering elements and the soil below, and then back again. Note that above the canopy, and in-between layers the radiation collapses to a node. This illustrates the point that radiation intensity at these points is "mixed", or in other words, horizontally homogeneous.

The process of connecting the boundary conditions of these scattering elements requires that we represent the share of horizontal space taken up by each, as a fraction of the total space f_A. Therefore, the mean radiant intensity between canopy layers, for both up-welling or down-welling radiation, will require a weighted sum of the intensity on the boundary of each scattering element, and potentially the boundary of any exposed ground. An example of a PPA canopy is provided in Figure 3. In this scenario, there are two plant functional types, where both are found in the upper layer, and both are found in the lower canopy layer. We assume that the scattering is continuous inside these elements following the conservation Equations 1 and the resulting analytical Solution 2. These elements can be connected in a system of equations, where the number of equations matches the number of unknown coefficients λ. In this example, there are 4 unique scattering elements, each with two λ terms, requiring a total of 8 equations.

Two equations connect the down-welling radiation coming from the atmosphere (constant over an interval of time, and known), one for each of the two scattering elements in the canopy. Two equations represent the boundary of mean down-welling radiation exiting the upper canopy, and entering each of the two pfts in the lower canopy. Two equations represent the mean up-welling radiation leaving the lower canopy and entering each of the two scattering elements in the upper canopy. And finally, we have two equations representing the boundary conditions between the two scattering elements in the lower canopy and the ground. This system will allow us to create an 8x8 matrix Ω, operated upon by vector Λ (i.e. the vector of λ) that equals a column vector with the additive terms Φ.

$$\Phi = \Omega \cdot \Lambda \tag{3}$$

In this system, we perform inversion to solve for vector Λ which then allows us to retrieve the scattering intensity as well as the absorption of scattering media anywhere in the system. We will use the following convention for indexing elements. Scattering elements are indexed by ascending PFT number, which are visually represented left to right, and then top down. When constructing balance equations, we will refer to elements in an upper canopy layer with index i within a set of elements I. We will refer to elements in the lower canopy layer with index j within a set J.

3.1 Solution for Upper Boundary with Atmosphere

The equation balancing down-welling diffuse radiation intensity from the atmosphere $R_{d,atm}$ parts. This is applied to both of the first two scattering elements, those in the upper canopy, $i = 1, 2$. Recall that the system of equations is really only solving for the diffuse components of radiation, and the complex terms already include the transmission and attenuation of beam radiation in the canopy.

$$R_{d,atm} = A_{dn(i)} + \lambda_{1(i)} B_{2(i)} + \lambda_{2(i)} B_{1(i)} \tag{4}$$

Or, rearranging so that the Φ term is on the left side of the equation:

$$R_{d,atm} - A_{dn(i)} = \lambda_{1(i)} \cdot B_{2(i)} + \lambda_{2(i)} \cdot B_{1(i)} \tag{5}$$

3.2 Solutions for Boundaries Between Canopy Layers

3.2.1 Down-welling

For the downward stream, the mean radiation coming from elements in set I in the layer above needs to balance with the mean radiation entering the top of any element j of the layer below. If there is a lower layer, then FATES assumes that the upper layer is closed (ie perfectly plastic) and the areas $f_{A(i)}$ should sum to unity. Note that the down-welling radiation exiting the upper layer elements is evaluated at $v = V$ (i.e. the bottom of the media), and the down-welling radiation at the top of the lower element is evaluated at the top of the media $v = 0$.

$$\begin{gathered} R_{dn(v=0,j)} = \sum^{I} f_{A(i)} R_{nd(v=V,i)} \\ A_{dn(j)} + \lambda_{1(j)} B_{2(j)} + \lambda_{2(j)} B_{1(j)} = \\ \sum^{I} f_{A(i)} \left(A_{dn(i)} e^{-\kappa_b V} + \lambda_{1(i)} B_{2(i)} e^{aV} + \lambda_{2(i)} B_{1(i)} e^{-aV} \right) \end{gathered} \tag{6}$$

This is expanded and partitioned into terms with and without λ. The left side is a component of Φ and the right side are components of Ω and Λ:

$$\begin{gathered} A_{dn(j)} - \sum^{I} f_{A(i)} A_{dn(i)} e^{-\kappa_b V} = \\ \sum^{I} f_{A(i)} \left(\lambda_{1(i)} \cdot B_{2(i)} e^{aV} + \lambda_{2(i)} \cdot B_{1(i)} e^{-aV} \right) - \lambda_{1(j)} \cdot B_{2(j)} - \lambda_{2(j)} \cdot B_{1(j)}) \end{gathered} \tag{7}$$

3.2.2 Up-welling

For the upward stream, the mean radiation coming from elements j in set J in the layer below, needs to balance with the radiation entering the top of any element i in the layer above. Note, there may be exposed ground reflecting light, however recall that a "ghost element" is applied with zero optical depth for open canopy areas at all layers. This element will transfer that un-obstructed light.

$$R_{up(v=V,i)} = \sum^{J} f_{A(j)} R_{up(v=0,j)} \tag{8}$$

Equation 8 can then be expanded by substituting in Equation 2, to generate balance Equation 9. This can then be organized into the components in Φ (left side) and Ω (right side), see Equation 10.

$$\begin{aligned} A_{up(i)}e^{-\kappa_b V} + \lambda_{1(i)}B_{1(i)}e^{aV} + \lambda_{2(i)}B_{2(i)}e^{-aV} = \\ \sum^{J}\left(f_{A(j)}A_{up(j)} + \lambda_{1(j)}B_{1(j)} + \lambda_{2(j)}B_{2(j)}\right) \end{aligned} \tag{9}$$

$$\begin{aligned} A_{up(i)}e^{-\kappa_b V} - \sum^{J}\left(f_{A(j)}A_{up(j)}\right) = \\ \sum^{J}\left(f_{A(j)} + \lambda_{1(j)}B_{1(j)} + \lambda_{2(j)}B_{2(j)}\right) - \lambda_{1(i)}B_{1(i)}e^{aV} - \lambda_{2(i)}B_{2(i)}e^{-aV} \end{aligned} \tag{10}$$

3.3 Solutions for Boundaries Between the Bottom Layer and the Ground

Here we form a flux balance between the lower edge of any scattering elements in the lowest layer of the canopy. For elements indexed j in the lowest canopy layer, the sum of the beam radiation at the bottom of the element times the ground beam albedo, and the down-welling diffuse radiation at the bottom of the element ($v = V$) is reflected by the diffuse ground albedo and should equal the up-welling radiation at the bottom of the same element $v = V$. The beam radiation intensity at the bottom of the element $R_{b(v=V,j)}$ is straightforward and can be derived from the trivial beam attenuation algorithm.

$$R_{up(v=V,j)} = \omega_{gd} R_{dn(v=V,j)} + \omega_{gb} R_{b(v=V,j)} \tag{11}$$

$$
\begin{aligned}
A_{up(j)}e^{-\kappa_b V} + \lambda_{1(j)}B_{1(j)}e^{aV} + \lambda_{2(j)}B_{2(j)}e^{-aV} = \\
\omega_{gd}\left(A_{dn(j)}e^{-\kappa_b V} + \lambda_{1(j)}\cdot B_{2(j)}e^{aV} + \lambda_{2(j)}\cdot B_{1(j)}e^{-aV}\right) + \\
\omega_{gb}R_{b(v=V,j)}
\end{aligned}
$$

$$
\begin{aligned}
A_{up(j)}e^{-\kappa_b V} - \omega_{gd}A_{dn(j)}e^{-\kappa_b V} - \omega_{gb}R_{b(v=V,j)} = \\
\omega_{gd}\left(\lambda_{1(j)}\cdot B_{2(j)}e^{aV} + \lambda_{2(j)}\cdot B_{1(j)}e^{-aV}\right) - \lambda_{1(j)}\cdot B_{1(j)}e^{aV} - \lambda_{2(j)}\cdot B_{2(j)}e^{-aV}
\end{aligned} \tag{12}
$$

4 Solving for Beam and Diffuse Driven Components Separately

The CESM and E3SM earth system models ask the land-model for a surface albedo, given the cosine of the zenith angle, vegetation canopy characteristics, soil characteristics and surface wetness, before the atmospheric model can provide the down-welling radiation conditions. **Therefore, we must solve the systems of equations, described in the previous subsections, with unit downwelling forcings, and then have the capacity to scale the results later in the time-step, when the actual magnitudes of the downwelling boundary conditions are known.**

To do this, we have to split the solution into two parts. We calculate two solutions independently, one for when there is only beam radiation ($R_{b,atm} = 1, R_{d,atm} = 0$), and for when there is only diffuse radiation ($R_{b,atm} = 0, R_{d,atm} = 1$), which leaves use with separate sets of solution constants for beam only (λ_b) and diffuse only (λ_d):

$$
\begin{aligned}
R_{up(v)} &= \overbrace{R_{b,atm}A_{up}e^{-\kappa_b v} + B_1e^{av}\lambda_{1,b} + B_2e^{-av}\lambda_{2,b}}^{\text{beam component}} + \overbrace{B_1e^{av}\lambda_{1,d} + B_2e^{-av}\lambda_{2,d}}^{\text{diffuse component}} \\
R_{dn(v)} &= R_{b,atm}A_{dn}e^{-\kappa_b v} + B_2e^{av}\lambda_{1,b} + B_1e^{-av}\lambda_{2,b} + B_2e^{av}\lambda_{1,d} + B_1e^{-av}\lambda_{2,d}
\end{aligned} \tag{13}
$$

Note that there is no A term for the diffuse only portion, as that term drops out when beam radiation is zero. Note that the unit downwelling diffuse term $R_{d,atm}$, is not explicitly cast in this representation. It is implicitly part of this solution, and introduced via the balance Equation (5), with a value of one for the diffuse side of the solution, and a value of zero for the beam side of the solution.

5 Sunlit Leaf Fractions

The sunlit fraction of leaves is related to the attenuation of beam radiation as it impinges upon the scattering elements vertically through the canopy. Recall, the intensity of beam radiation, given an upper boundary flux of R_{b0} uses Beer's law of exponential decrease, using a beam optical depth $-\kappa_b$ and total depth integrated vegetation area index v:

$$\frac{R_{b(v)}}{R_{b0}} = e^{-\kappa_b v} \tag{14}$$

This is consistent with the function that defines the sunlit fraction f_{sun} of the scattering media at depth v (also see NCAR Technical Note):

$$f_{sun(v)} = e^{-\kappa_b v} \tag{15}$$

For FATES, we need to generalize this formulation to incorporate a canopy layer above, and the mean sunlit fraction over a depth of interest v_1 to v_2. First, lets define the depth as the depth at the top of the layer v_{top}, and the delta from that top v.

$$\begin{aligned} f_{sun(v)} &= e^{-\kappa_b (v_{top}+v)} \\ f_{sun(v)} &= e^{-\kappa_b v_{top}} e^{-\kappa_b v} \end{aligned} \tag{16}$$

The first term in the in Equation 16 can alternately be defined as the ratio of the beam radiation $R_{b0(c)}$ at the top of canopy layer c , relative to the incident radiation intensity at the top of the canopy $R_{b,atm}$:

$$f_{sun(c,v)} = \frac{R_{b0(c)}}{R_{b,atm}} e^{-\kappa_b v} \tag{17}$$

To get the mean sunlit fraction, we integrate from our points of interest and divide by the difference.

$$\overline{f}_{sun(c,v_1-v_2)} = \frac{R_{b0(c)}}{R_{b,atm}} \frac{1}{v_2 - v_1} \int_{v_1}^{v_2} e^{-\kappa_b v} dv \tag{18}$$

$$\overline{f}_{sun(c,v_1-v_2)} = \frac{R_{b0(c)}}{R_{b,atm}} \frac{e^{-\kappa_b v_1} - e^{-\kappa_b v_2}}{\kappa_b(v_2 - v_1)} \tag{19}$$

6 Appendix

6.1 Full Derivation of the Analytical Generalized Solution

Starting with the conservation equation, see Equation 1:

$$\begin{aligned}
\frac{dR_{dn}}{dv} &= -\kappa_d(1-(1-\beta_d)\omega)R_{dn} + \kappa_d\beta_d\omega R_{up} + \kappa_b\omega(1-\beta_b)R_{b0}e^{-\kappa_b v} \\
\frac{dR_{up}}{dv} &= \kappa_d(1-(1-\beta_d)\omega)R_{up} - \kappa_d\beta_d\omega R_{dn} - \kappa_b\omega\beta_b R_{b0}e^{-\kappa_b v}
\end{aligned} \tag{20}$$

The first step is to and recast the equations as R_- and R_+:

$$\begin{gathered}
R_+ = R_{up} + R_{dn}, \qquad R_- = R_{dn} - R_{up} \\
\frac{dR_+}{dv} = \frac{dR_{up}}{dv} + \frac{dR_{dn}}{dv}, \qquad \frac{dR_-}{dv} = \frac{dR_{dn}}{dv} - \frac{dR_{up}}{dv}
\end{gathered} \tag{21}$$

$$\begin{aligned}
\frac{dR_+}{dv} = &\left[\kappa_d(1-(1-\beta_d)\omega)R_{up} - \kappa_d\beta_d\omega R_{dn} - \kappa_b\omega\beta_b R_{b0}e^{-\kappa_b v}\right] + \\
&\left[-\kappa_d(1-(1-\beta_d)\omega)R_{dn} + \kappa_d\beta_d\omega R_{up} + \kappa_b\omega(1-\beta_b)R_{b0}e^{-\kappa_b v}\right] \\
\frac{dR_-}{dv} = &\left[-\kappa_d(1-(1-\beta_d)\omega)R_{dn} + \kappa_d\beta_d\omega R_{up} + \kappa_b\omega(1-\beta_b)R_{b0}e^{-\kappa_b v}\right] \\
&- \left[\kappa_d(1-(1-\beta_d)\omega)R_{up} - \kappa_d\beta_d\omega R_{dn} - \kappa_b\omega\beta_b R_{b0}e^{-\kappa_b v}\right]
\end{aligned} \tag{22}$$

Moving terms around and re-combing the R_{up} and R_{dn}:

$$\begin{aligned}
\frac{dR_+}{dv} &= -\kappa_d\left[1-\omega+2\beta_d\omega\right]R_- + \kappa_b\left[\omega - 2\omega\beta_b\right]R_{b0}e^{-\kappa_b v} \\
\frac{dR_-}{dv} &= -\kappa_d\left[1-\omega\right]R_+ + \kappa_b\omega R_{b0}e^{-\kappa_b v}
\end{aligned} \tag{23}$$

Now that each equation has only one term, we can differentiate to get the second derivative:

$$
\begin{aligned}
\frac{d^2R_+}{dv^2} =& -\kappa_d \left[1-\omega+2\beta_d\omega\right] \frac{dR_-}{dv} - \kappa_b^2 \left[\omega - 2\omega\beta_b\right] R_{b0}e^{-\kappa_b v} \\
\frac{d^2R_-}{dv^2} =& -\kappa_d \left[1-\omega\right] \frac{dR_+}{dv} - \kappa_b^2 \omega R_{b0}e^{-\kappa_b v}
\end{aligned}
\tag{24}
$$

Replace the first order differential terms with their expansions, and do some algebra:

$$
\begin{aligned}
\frac{d^2R_+}{dv^2} =& \kappa_d^2 \left(1-\omega\right) \left(1-\omega+2\beta_d\omega\right) R_+ - \left[\kappa_d \left(1-\omega+2\beta_d\omega\right) + \kappa_b(1-2\beta_b)\right] \kappa_b\omega R_{b0}e^{-\kappa_b v} \\
\frac{d^2R_-}{dv^2} =& \kappa_d^2 \left(1-\omega\right) \left(1-\omega+2\beta_d\omega\right) R_- - \left[\kappa_d \left(1-\omega\right)\left(1-2\beta_b\right) + \kappa_b\right] \kappa_b\omega R_{b0}e^{-\kappa_b v}
\end{aligned}
\tag{25}
$$

Now we have two expressions where the second order differential is only dependent on its self and a constant, for which there is an analytical solution with unknown constants λ_1 and λ_2, given the form:

$$
\begin{aligned}
\frac{d^2y}{dx^2} =& ay + be^{-cx} \\
y_{(x)} =& \frac{be^{-cx}}{c^2-a} + \lambda_1 e^{\sqrt{a}x} + \lambda_2 e^{-\sqrt{a}x}
\end{aligned}
\tag{26}
$$

We can then cast Equation 25 in this form, noting to give them unique constants.

$$
\begin{aligned}
a =& \sqrt{\kappa_d^2 \left(1-\omega+2\beta_d\omega\right)\left(1-\omega\right)} \\
b_1 =& -\left[\kappa_d \left(1-\omega+2\beta_d\omega\right) + \kappa_b(1-2\beta_b)\right] \kappa_b\omega R_{b0} \\
b_2 =& -\left[\kappa_d \left(1-\omega\right)\left(1-2\beta_b\right) + \kappa_b\right] \kappa_b\omega R_{b0} \\
R_+ =& \frac{b_1 e^{-\kappa_b v}}{\kappa_b^2 - a^2} + \lambda_1 e^{av} + \lambda_2 e^{-av} \\
R_- =& \frac{b_2 e^{-\kappa_b v}}{\kappa_b^2 - a^2} + \lambda_3 e^{av} + \lambda_4 e^{-av}
\end{aligned}
\tag{27}
$$

We have two analytical solutions, yet we have four unkown constants (λ's). In the next steps, we calculate the derivatives of the analytical solutions, and then compare them against the original transformed conservation equations (23), to find equivalence of the λ terms in the R_+ and R_- equations.

$$\begin{aligned}
\frac{d}{dv}\left[R_+\right] =& \frac{-\kappa_b b_1 e^{-\kappa_b v}}{\kappa_b^2 - a^2} + \lambda_1 a e^{av} - \lambda_2 a e^{-av} \\
\frac{d}{dv}\left[R_-\right] =& \frac{-\kappa_b b_2 e^{-\kappa_b v}}{\kappa_b^2 - a^2} + \lambda_3 a e^{av} - \lambda_4 a e^{-av}
\end{aligned} \tag{28}$$

Note that after the substitutions have been made, there are three different components of the equations, the terms not associated with any λ's (the intercept), the terms associated with λ_1 and λ_3 attached to e^{av}, and the terms associated with λ_2 and λ_4 attached to the e^{-av}. The λ_3 and λ_4 terms can be substituted by expressions with respect to λ_1 and λ_2, respectively, via a term ν:

$$\begin{aligned}
\nu =& \sqrt{\frac{(1-\omega)}{(1-\omega+2\beta_d\omega)}} \\
\lambda_3 =& -\nu\lambda_1 \\
\lambda_4 =& \nu\lambda_2
\end{aligned} \tag{29}$$

(**An important note about this previous step**: In order to simplify the ν term, we perform an algebraic manipulation where we square the denominator and the numerator, and then take the square root of the whole fraction. This manipulation is only possible if the fraction has only positive terms for both numerator and denominator. Those who are reproducing this result, make sure this is so! Notice in the result here, that this is true, because both ω and β_d are by definition, always positive and less than 1.)

The analytical solution for the transformed coordinates can then be recast with just two unknown terms:

$$\begin{aligned}
R_+ =& \frac{b_1 e^{-\kappa_b v}}{\kappa_b^2 - a^2} + \lambda_1 e^{av} + \lambda_2 e^{-av} \\
R_- =& \frac{b_2 e^{-\kappa_b v}}{\kappa_b^2 - a^2} - \nu\lambda_1 e^{av} + \nu\lambda_2 e^{-av}
\end{aligned} \tag{30}$$

The analytical solution can then be re-cast on the original coordinates, R_{up} and R_{dn}. Here is the reverse transformation back to those coordinates.

$$
\begin{aligned}
R_{up} &= \tfrac{1}{2}(R_{+} - R_{-}) \\
R_{dn} &= \tfrac{1}{2}(R_{+} + R_{-})
\end{aligned}
\tag{31}
$$

The result, Equation 32, is defined by various known constants (some represented by combined terms) and the two unknown constants λ, which sets up various methods of solution where the basis has two equations and two unknowns.

$$
\begin{aligned}
R_{up} &= \overbrace{\frac{1}{2}\frac{b_1 - b_2}{\kappa_b^2 - a^2}}^{A_{up}\text{ term}} e^{-\kappa_b v} + \lambda_1 \overbrace{\frac{1}{2}(1+\nu)}^{B_1\text{ term}} e^{av} + \lambda_2 \overbrace{\frac{1}{2}(1-\nu)}^{B_2\text{ term}} e^{-av} \\
R_{dn} &= \overbrace{\frac{1}{2}\frac{b_1 + b_2}{\kappa_b^2 - a^2}}^{A_{dn}\text{ term}} e^{-\kappa_b v} + \lambda_1 \overbrace{\frac{1}{2}(1-\nu)}^{B_2\text{ term}} e^{av} + \lambda_2 \overbrace{\frac{1}{2}(1+\nu)}^{B_1\text{ term}} e^{-av}
\end{aligned}
\tag{32}
$$

This form of the two-stream analytical solution can then be compared with the more simplified form shown earlier in this document, Equation 2.

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